Cohesive Subgraph Computation over Large Sparse Graphs

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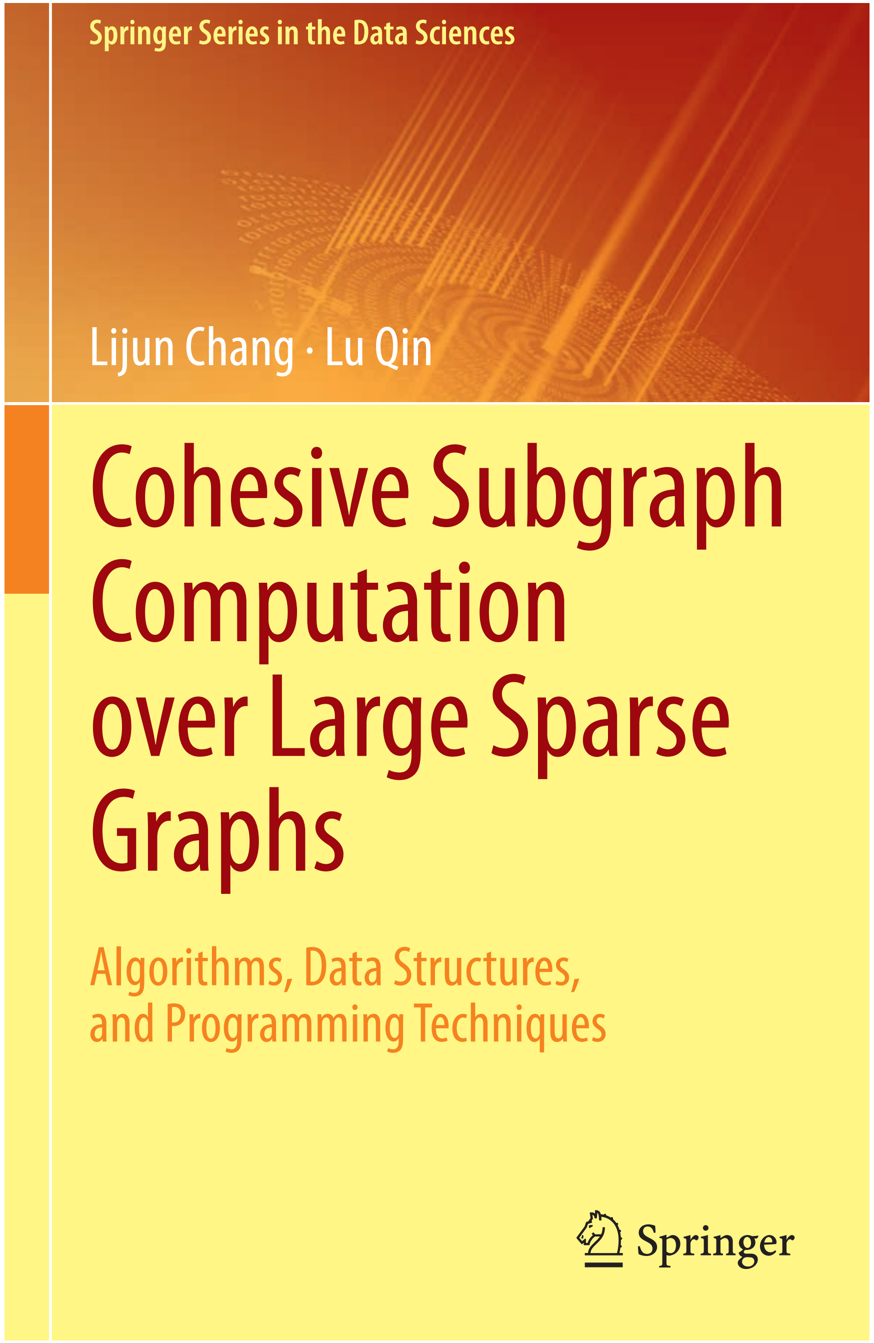
Slides: lijunchang.github.io/icde19_tutorial.pdf

ICDE
MACAU 2019



Springer Series in the Data Sciences

Lijun Chang · Lu Qin

Cohesive Subgraph Computation over Large Sparse Graphs

Algorithms, Data Structures, and Programming Techniques

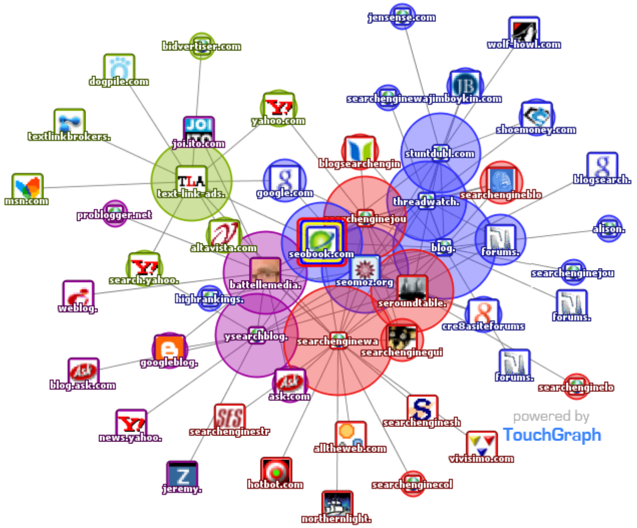
Springer

Outline

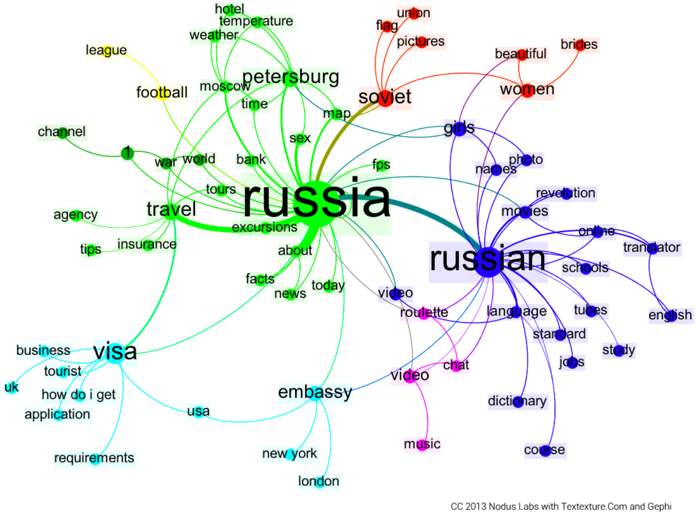
- Background
- Core Decomposition
- Densest Subgraph Computation
- Higher-order Dense Subgraph Computation
- Future Directions

Graphs are Everywhere

Social networks



Web graphs

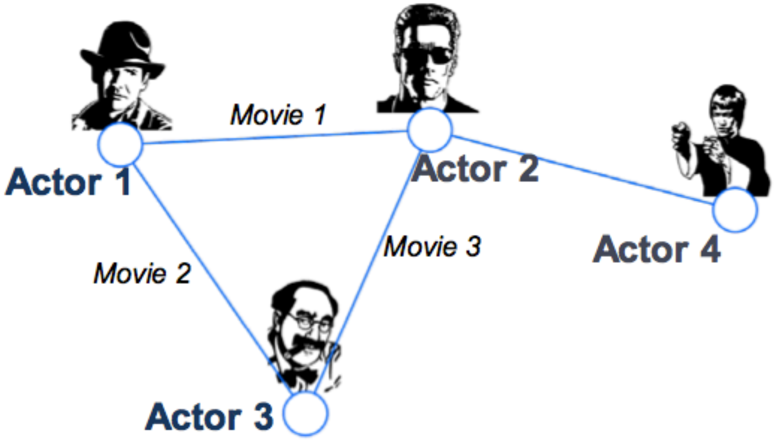


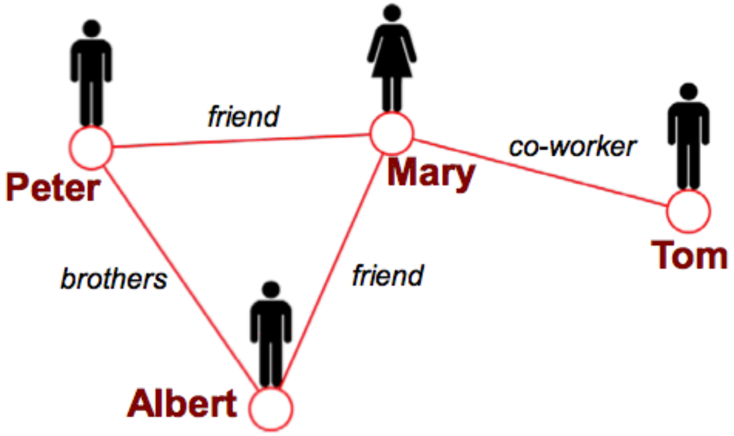
Graph of texts

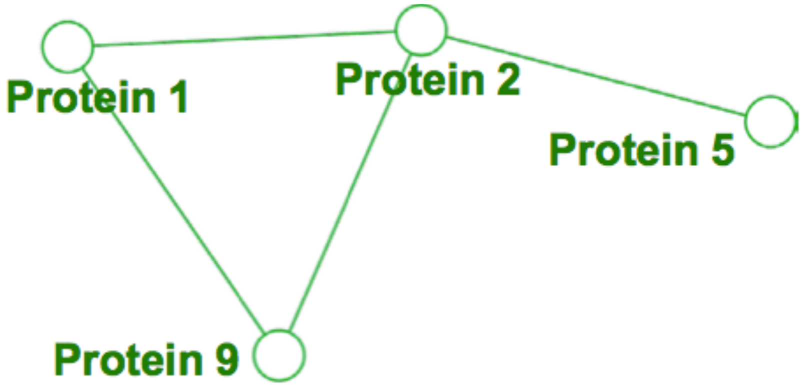
Internet of things

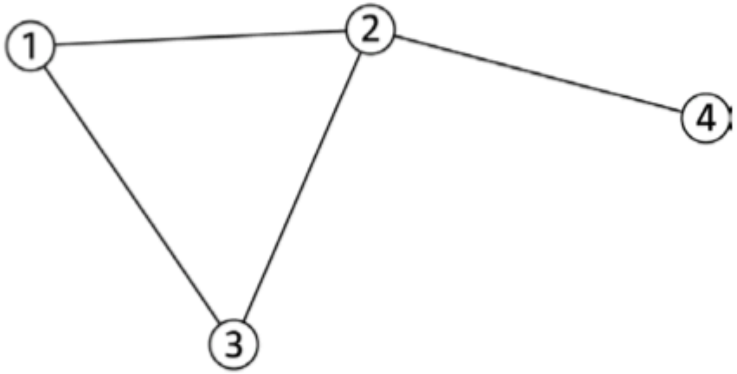
Graph Model is Simple

- A graph *G(V, E)* consists of a set of vertices *V* and a set of edges *E*





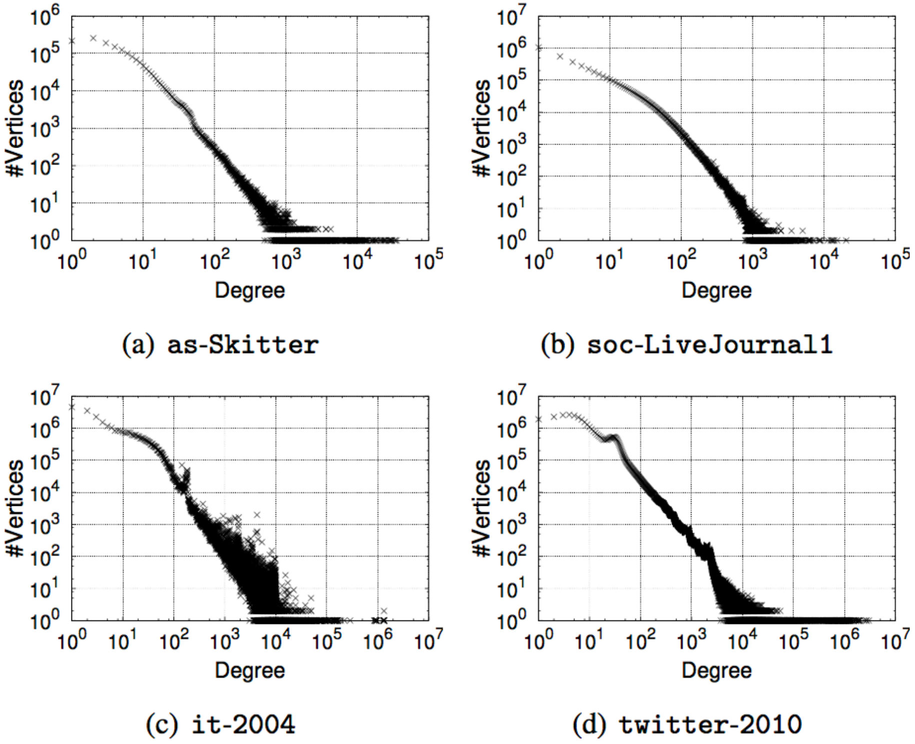




We are interested in analyzing the topological structure of real graphs!

Real Graphs are not Random Graphs

- Real graphs are not random graphs (e.g., the Erdos-Renyi random graph model), but have fascinating patterns and properties.
  - The degree distribution is skewed, following a power-law



(a) as-Skitter

(b) soc-LiveJournal1

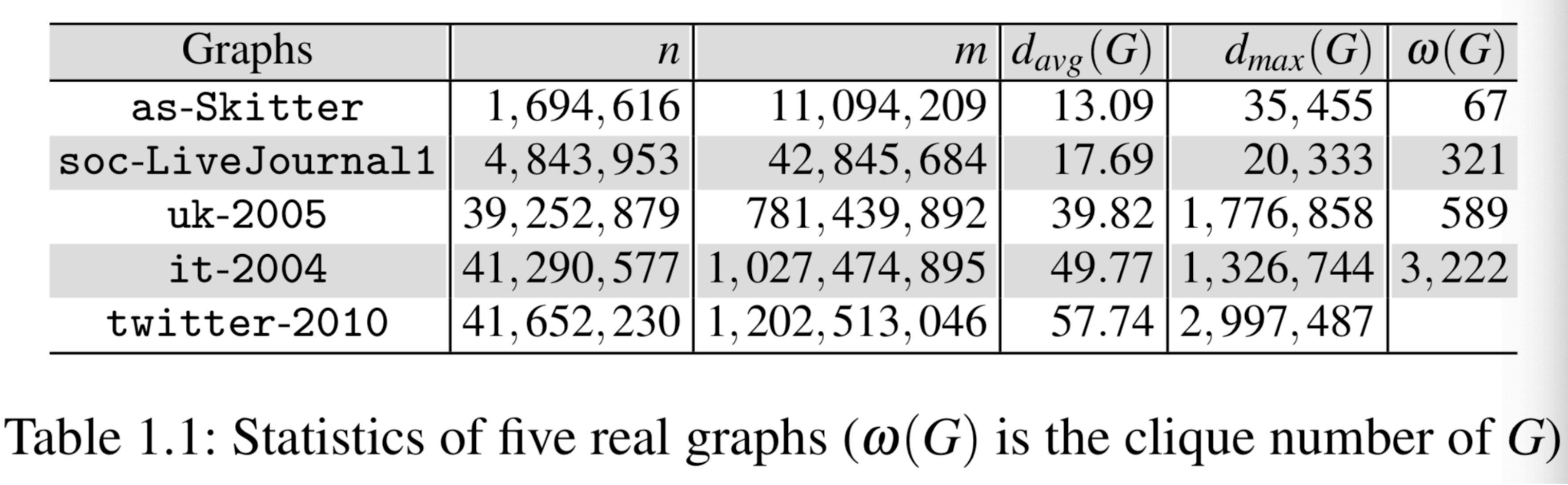
(c) it-2004

(d) twitter-2010

Fig. 1.2: Degree distributions

Real Graphs are not Random Graphs

- Real graphs are not random graphs (e.g., the Erdos-Renyi random graph model), but have fascinating patterns and properties.
  - Real graphs are globally sparse but locally dense
    - The entire graph is sparse, but there are groups of vertices with high concentration of edges within them



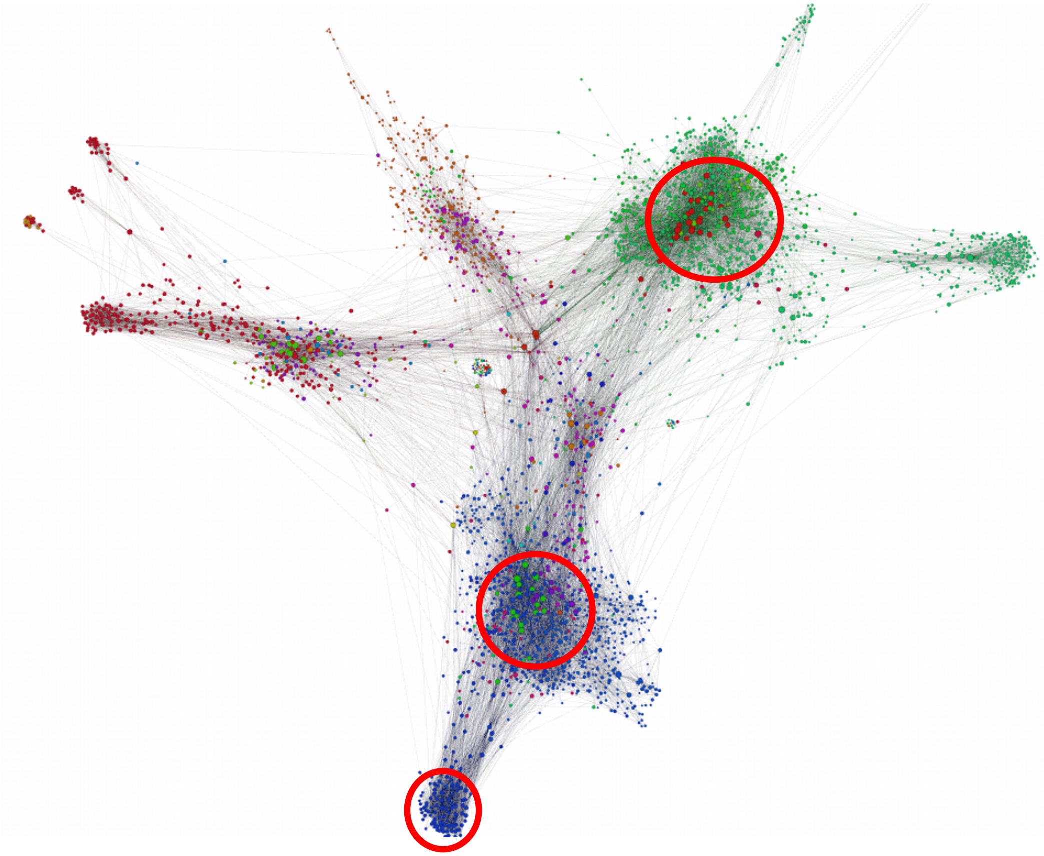
Graphs	n	m	$d_{avg}(G)$	$d_{max}(G)$	$\omega(G)$
`as-Skitter`	1,694,616	11,094,209	13.09	35,455	67
`soc-LiveJournal1`	4,843,953	42,845,684	17.69	20,333	321
`uk-2005`	39,252,879	781,439,892	39.82	1,776,858	589
`it-2004`	41,290,577	1,027,474,895	49.77	1,326,744	3,222
`twitter-2010`	41,652,230	1,202,513,046	57.74	2,997,487	

Table 1.1: Statistics of five real graphs ($\omega(G)$ is the clique number of G)

We are interested in finding "dense" subgraphs from large real graphs!

Informal Problem Definition

- Given a large sparse graph (e.g., social network, communication network, information network, biological network), find subgraphs that are densely connected or build a hierarchical structure for all dense subgraphs.



Applications of Finding Dense Subgraphs

- It has applications in any context that information can be encoded as a graph

- For example, dense subgraphs correspond to
  - Communities in *social networks*
  - Groups of web pages dealing with the same or related topics in *World Wide Web*
  - Groups of proteins having the same specific function within the cell in *biology*
  - Functional modules such as cycles and pathways in *metabolic networks*
  - Compartments in *food webs*
  - Stories in *twitter data*
  - ……

Focus of this Tutorial

- In this tutorial, we mainly focus on the fundamental technical developments of efficient dense subgraph computation
  - Efficiency is an important issue when analysing large graphs

Where to Find Large Real Graphs?

- Stanford Network Analysis Project (SNAP) [Leskovec and Krevl 2014]
  - From medium to large graphs. It includes social networks, web graphs, road networks, internet networks, citation networks, collaboration networks, and communication networks.
  - com-Friendster: 65 million vertices, 1.8 billion edges.
- Laboratory for Web Algorithmics (LAW) [Boldi and Vigna 2004]
  - Large graphs with size up to 1 billion vertices and tens of billions of edges. The networks are mainly web graphs and social networks.
  - eu-2015: 1 billion vertices, 91 billion edges.
- Network Repository [Rossi and Ahmed 2015]
  - Thousands of graphs with up to billions of vertices and tens of billions of edges.

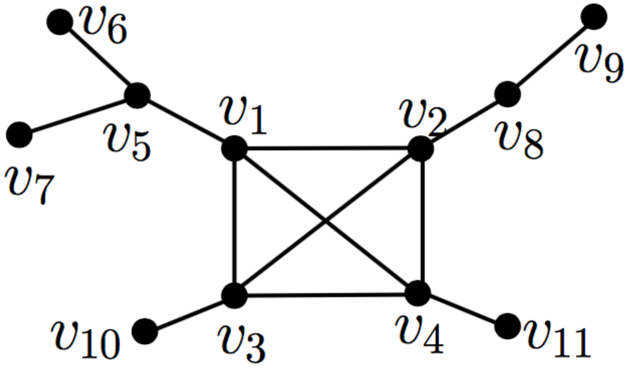
How to Store Large Sparse Graph in Memory?

- Graph representation
  - Adjacency Matrix ✗
    - Cannot store graph with over 10^5 vertices
  - Adjacency Lists ✗
    - Better, but requires *$4m$* integers
  - Adjacency Array or Compressed Sparse Row (**CSR**) ✓
    - Represents an undirected graph by *$2m+n+O(1)$* integers

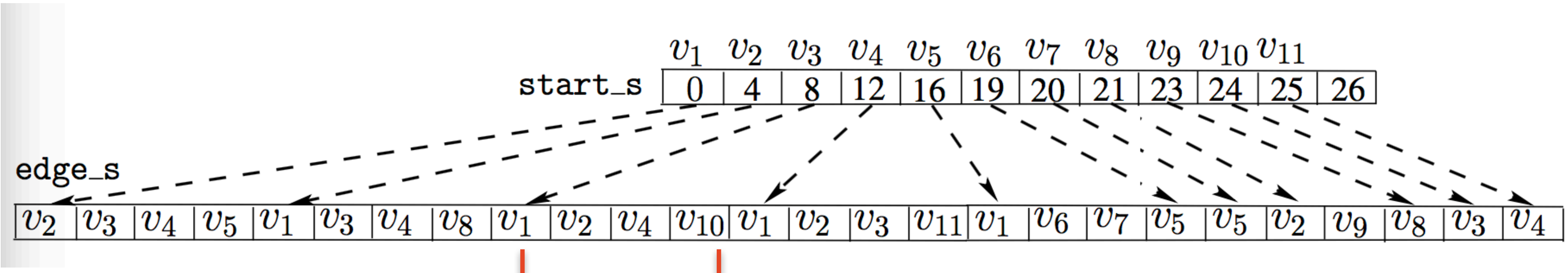
n: the number of vertices
m: the number of undirected edges

The Adjacency Array (CSR) Representation

- An example graph



- Its adjacency array representation



start_s

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	
0	4	8	12	16	19	20	21	23	24	25	26

edge_s

v_2	v_3	v_4	v_5	v_1	v_3	v_4	v_8	v_1	v_2	v_4	v_{10}	v_1	v_2	v_3	v_{11}	v_1	v_6	v_7	v_5	v_5	v_2	v_9	v_8	v_3	v_4

Neighbors of v_3

Cohesive/Dense Subgraph Computation

- Given a graph $G = (V, E)$ with vertices V and edges $E \subseteq V \times V$, we aim to efficiently compute dense subgraphs in G.
  - Either compute the subgraph with the highest density, or compute all (maximal) subgraphs whose density are larger than a threshold (e.g., k)
  - $n = |V|$
  - $m = |E|$

- How to measure the density of a (sub)-graph?
  - Edge ratio (2m/(n(n-1))): ratio of the number of edges to the maximum possible number of edges
    - However, small graphs usually have higher edge ratio. E.g., triangle
  - Average degree (2m/n)
  - Minimum degree

Cohesiveness Measures

- Minimum degree: core decomposition
  - Minimum number of edges each vertex participates in

- Average degree: densest subgraph
  - Average number of edges each vertex participates in

- Higher order
  - Minimum number of triangles each edge participates in: truss decomposition
  - Average number of k-cliques each vertex participates in: k-clique densest subgraph

- Edge connectivity
- …

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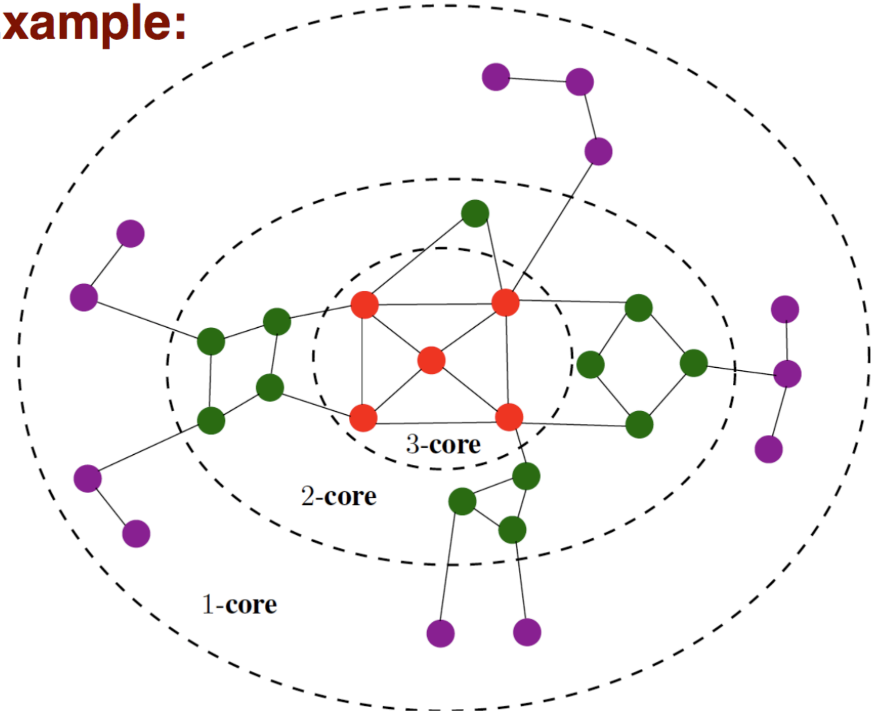
Core Decomposition

- *k*-core: the maximal subgraph in which every vertex has degree at least *k* within the subgraph
- Core number *core(u)* of a vertex: the largest *k* for which the *k*-core contains the vertex

Example:



$G_0 = G$

$G_1 = \text{1-core of } G$

$G_2 = \text{2-core of } G$

$G_3 = \text{3-core of } G$

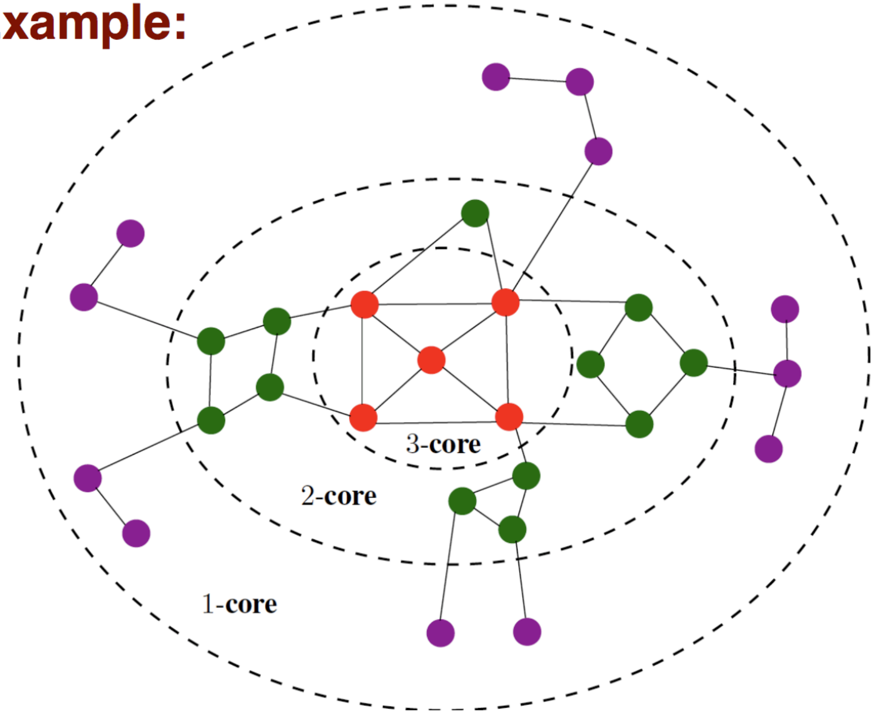
$G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3$

[Malliaros et al 2016]

Core Decomposition

- Core decomposition: compute the core numbers of all vertices
  - *k*-core is the subgraph induced by all vertices with core numbers at least *k*

Example:



🟣 Core number $c_i = 1$

🟢 Core number $c_i = 2$

🔴 Core number $c_i = 3$

$G_0 = G$

$G_1 = 1\text{-core of } G$

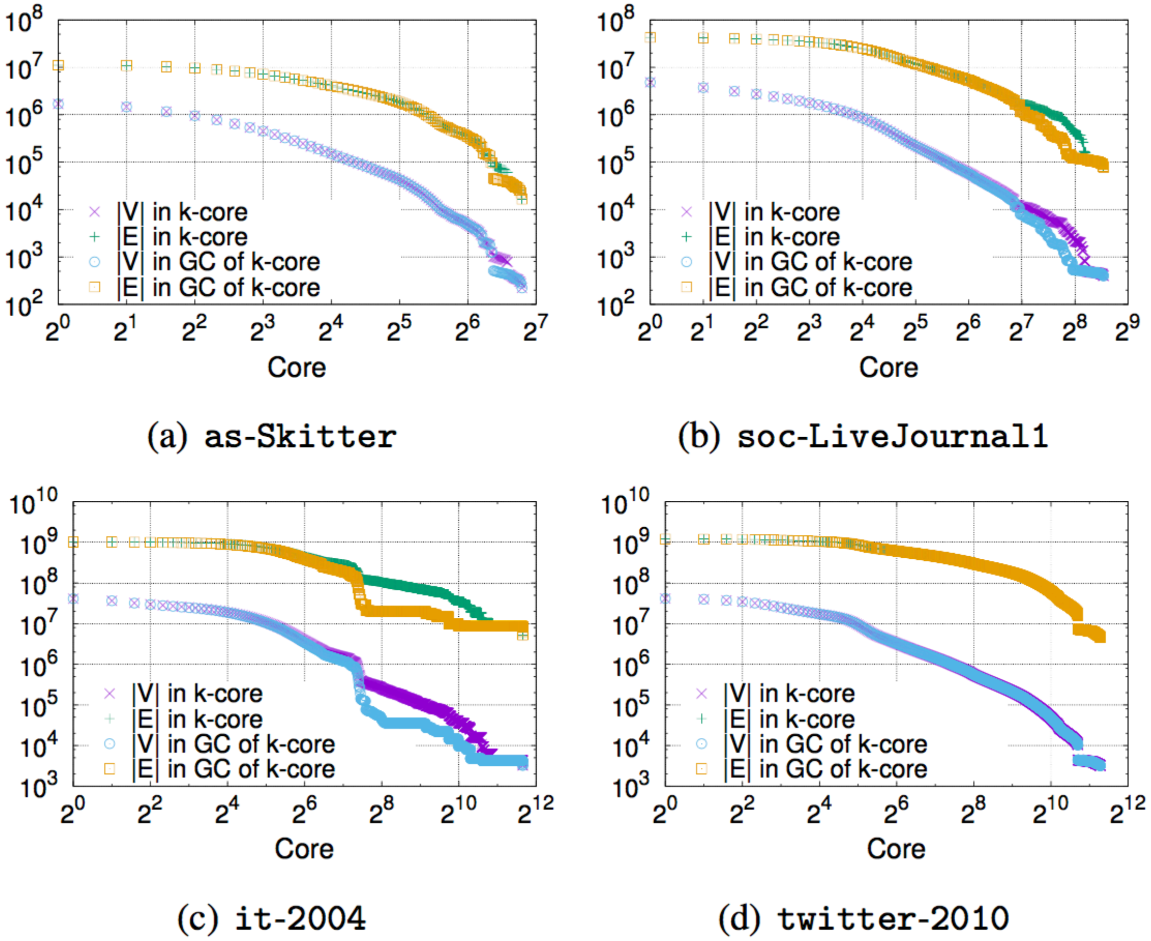
$G_2 = 2\text{-core of } G$

$G_3 = 3\text{-core of } G$

$G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3$

[Malliaros et al 2016]

K-core size Distribution



(a) as-Skitter

(b) soc-LiveJournal1

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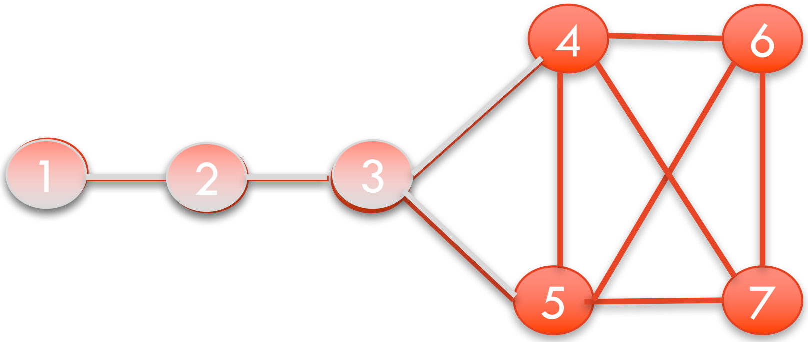
(d) twitter-2010

Fig. 3.2: Number of vertices and edges in (Giant Component of) *k*-core (varying *k*)

The Peeling Algorithm

- Basic idea for computing *k*-core: iteratively remove all vertices whose degree are smaller than *k*.
  - Core decomposition: iterate the process for *k* values increasing from *1*



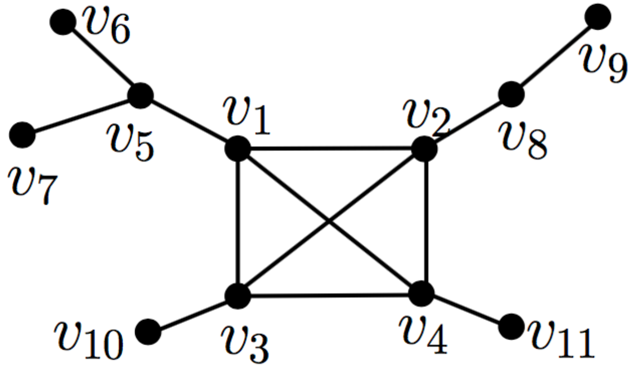
3-core

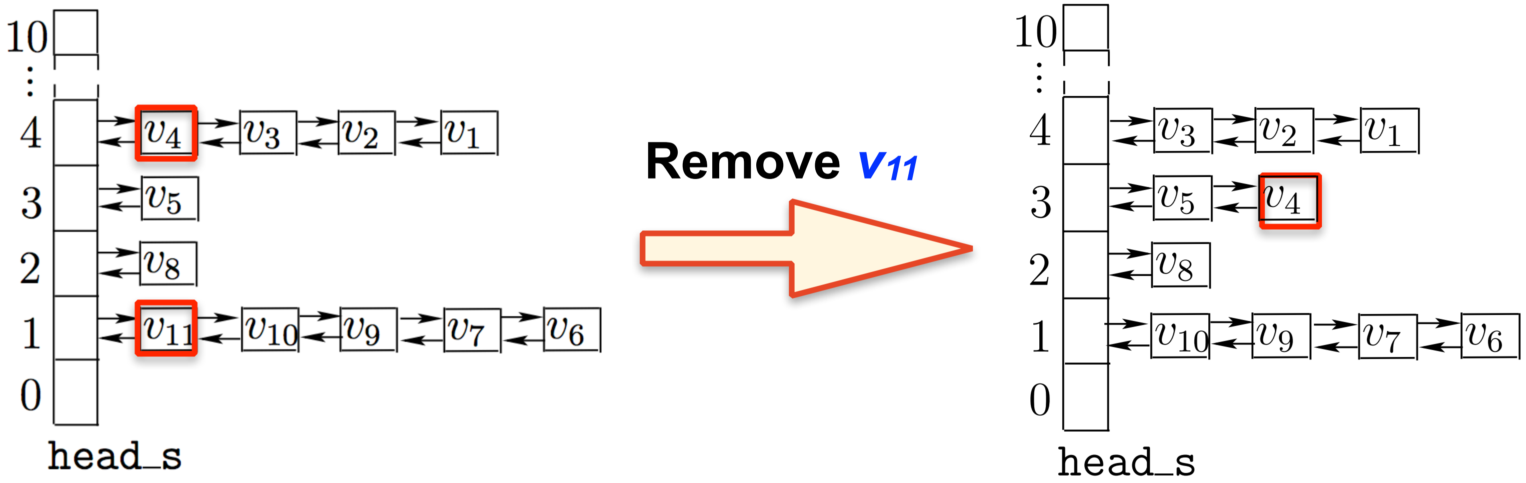
2-core

- Naively going though all vertices to find a vertex of degree smaller than *k* in each iteration will result in $O(n^2)$ time algorithm

A Linear-time Implementation

- Using a data structure to dynamically maintain the vertices of a specific degree, results in O(*m*) algorithm





Remove v_{11}

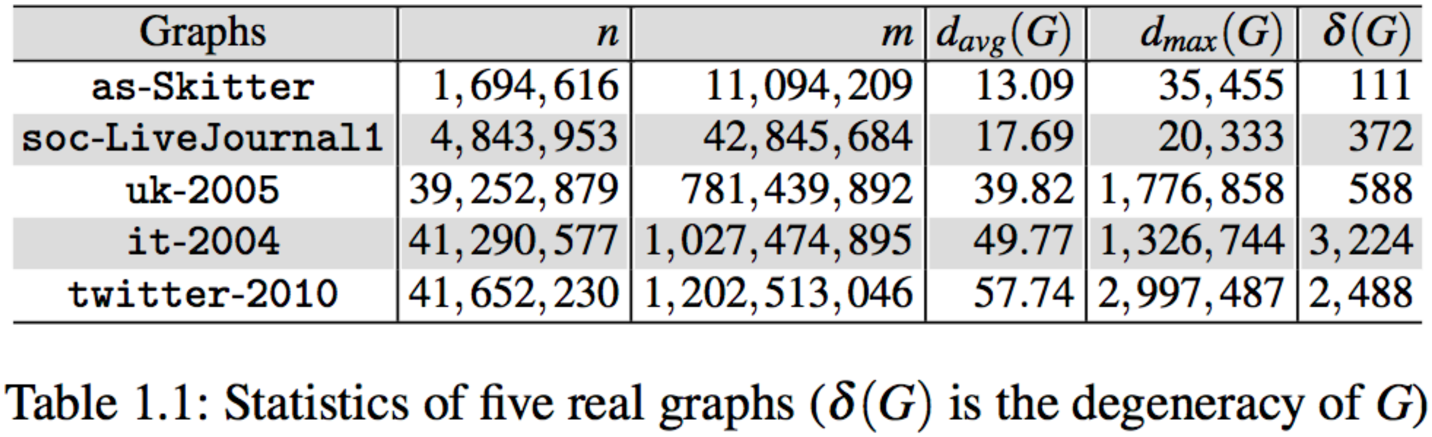
The Peeling Algorithm

- To compute k-core, we can remove an arbitrary vertex among all vertices of degree smaller than k.

- In practice, the peeling algorithm usually refers to the algorithm that iteratively removes the vertex with the smallest degree.
    - The previous data structure still can implement this algorithm to run in $O(m)$ time.

Other Applications of the Peeling Algorithm

- It computes the degeneracy $\delta(G)$ of a graph G
  - $\delta(G)$ is the maximum value among the minimum vertex degrees of all subgraphs of G
    - Each subgraph of G has a vertex with small degree (i.e., $\leq \delta(G)$)
    - There exists a subgraph with minimum degree $\delta(G)$
  - $\delta(G)$ equals the largest core number in core decomposition
  - $\delta(G) \leq \lceil \sqrt{2m+n} \rceil$
  - $\delta(G)$ measures how sparse a graph is



Graphs	n	m	$d_{avg}(G)$	$d_{max}(G)$	$\delta(G)$
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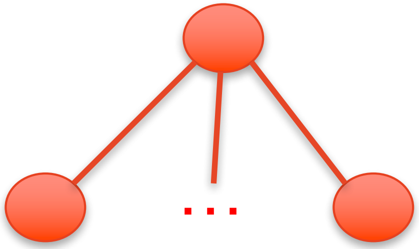
uk-2005	39,252,879	781,439,892	39.82	1,776,858	588
it-2004	41,290,577	1,027,474,895	49.77	1,326,744	3,224
twitter-2010	41,652,230	1,202,513,046	57.74	2,997,487	2,488

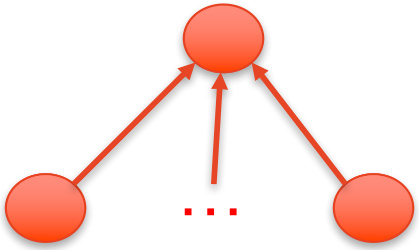
Table 1.1: Statistics of five real graphs ($\delta(G)$ is the degeneracy of G)

Other Applications of the Peeling Algorithm

- It computes the degeneracy $\delta(G)$ of a graph G
- It computes a degeneracy ordering of vertices of G
  - A permutation $(v_1, v_2, \ldots, v_n)$ of all vertices of G is a degeneracy ordering of G if every vertex v_i has the minimum degree in the subgraph induced by $\{v_i,\ldots,v_n\}$.
  - If we orient the graph according to a degeneracy ordering, then the maximum out-degree of the resulting directed graph is $\delta(G)$



Maximum degree: *n-1*



Maximum out-degree: *1*

Other Applications of the Peeling Algorithm

- It computes the degeneracy $\delta(G)$ of a graph G
- It computes a degeneracy ordering of vertices of G
- It computes an approximate value for the arboricity $\alpha(G)$ of a graph G
  - $\alpha(G)$ is the minimum number of forests needed to cover all edges of a graph
  - $\alpha(G)$ is frequently used in analyzing time complexities of algorithms, especially triangle enumeration/counting related algorithms
  - Degeneracy $\delta(G)$ tightly bounds the arboricity $\alpha(G)$ of a graph: $\alpha(G) \leq \delta(G) < 2\times\alpha(G)$

Other Applications of the Peeling Algorithm

- It computes the degeneracy $\delta(G)$ of a graph G
- It computes a degeneracy ordering of vertices of G
- It computes an approximate value for the arboricity $\alpha(G)$ of a graph G
- It computes an approximate solution to the densest subgraph (will be covered later)

H-index-based Local Algorithm

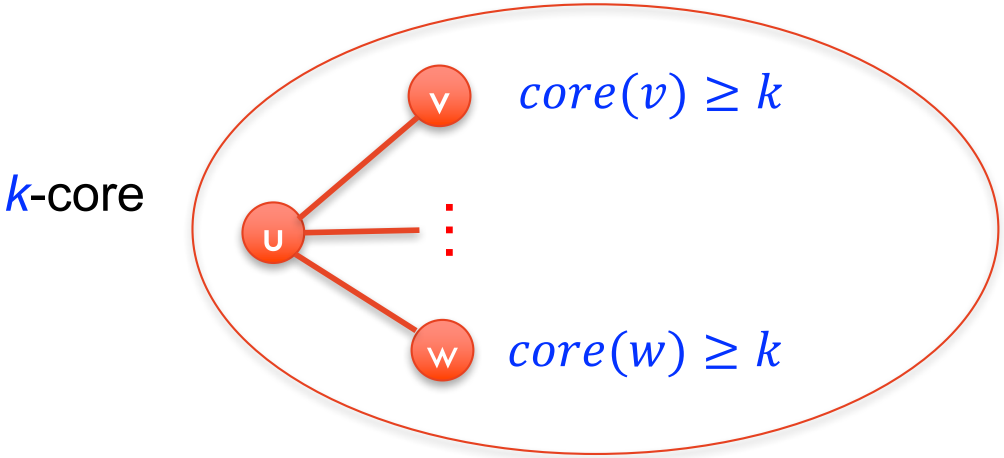
- The peeling algorithm is inherently sequential, and has limited parallelizability.

- There is an H-index-based local algorithm that works well in practice for different settings
  - e.g., parallel setting, distributed setting, I/O-efficient setting, in-memory

- Given a multi-set *S* of positive numbers, *h-index(S)* is the largest integer *k* such that $|\{s \in S : s \geq k\}| \geq k$
  - E.g. h-index({1,1,1,1}) = 1
  - h-index({4,3,2,1}) = 2

H-index-based Local Algorithm

- Fact 1: let $C_u = \{core(v): v \in N(u)\}$, then $core(u) = h{-}index(C_u)$
  - Let $\mathrm{k} = core(u)$, u must have at least *k* neighbors in the *k*-core.



- Fact 2: let $\overline{core}(v)$ be an upper bound of $core(v)$ and $\bar{C}_u = \{\overline{core}(v): v \in N(u)\}$, then $h{-}index(\bar{C}_u)$ is an upper bound of $core(u)$

H-index-based Local Algorithm

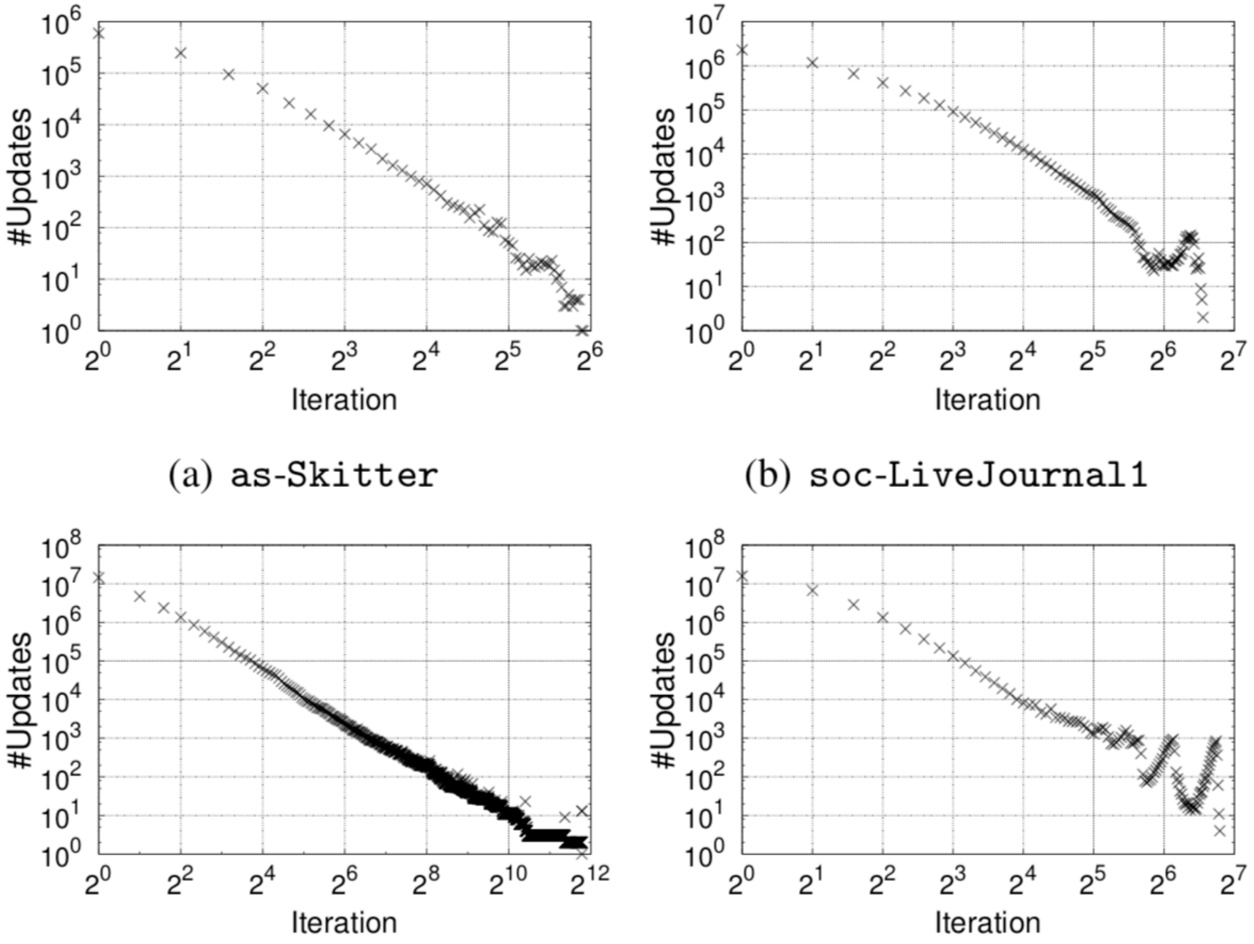
- Fact 1: let $C_u = \{core(v): v \in N(u)\}$, then $core(u) = h\text{−}index(C_u)$
- Fact 2: let $\bar{C}_u = \{\overline{core}(v): v \in N(u)\}$, then $core(u) \leq h\text{−}index(\bar{C}_u)$

- Algorithm:
  - Initialize $\overline{core}(v)$ to be the degree of v for all vertices
  - Repeat until converge: reassign $\overline{core}(u)$ as $h\text{−}index(\bar{C}_u)$ for all vertices

- h-index is monotone
  - The upper bounds cannot increase
  - The upper bounds converge to the true core numbers.

- Optimization: do not need to update the upper bound for every vertex in each iteration

H-index-based Local Algorithm



(a) as-Skitter

(b) soc-LiveJournal1

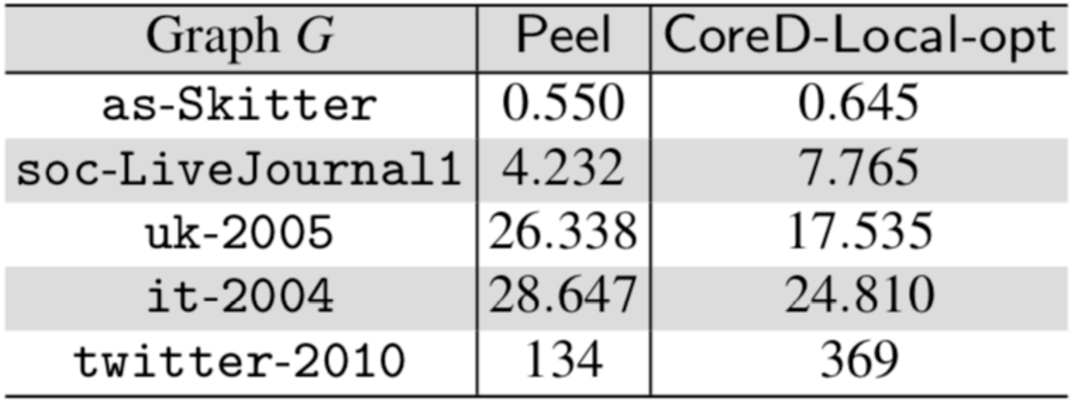
(c) it-2004

(d) twitter-2010

Fig. 3.5: Number of updates in different iterations

H-index-based Local Algorithm

- Empirical in-memory running time comparison (in seconds)



Graph G	Peel	CoreD-Local-opt
as-Skitter	0.550	0.645
soc-LiveJournal1	4.232	7.765
uk-2005	26.338	17.535
it-2004	28.647	24.810
twitter-2010	134	369

- The running time highly depends on the processing order of vertices
  - E.g., if processing vertices in the degeneracy ordering, then the time complexity is linear

Other Works on Core Decomposition

- Core decomposition for dynamic graph
  - How to maintain the core number when graph changes?
  - [Zhang et al 2017]

- Core decomposition for uncertain graph
  - [Bonchi et al 2014]

- Core decomposition for directed graph
  - [Giatsidis et al 2011]

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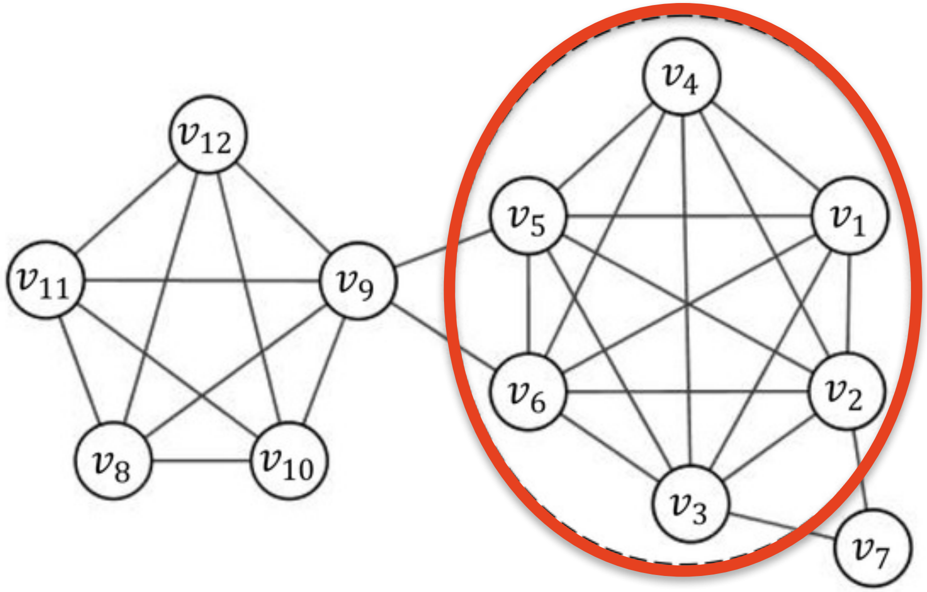
Densest Subgraph

- Find the subset S of vertices in G where the induced subgraph of G by S has the largest average degree among all subsets



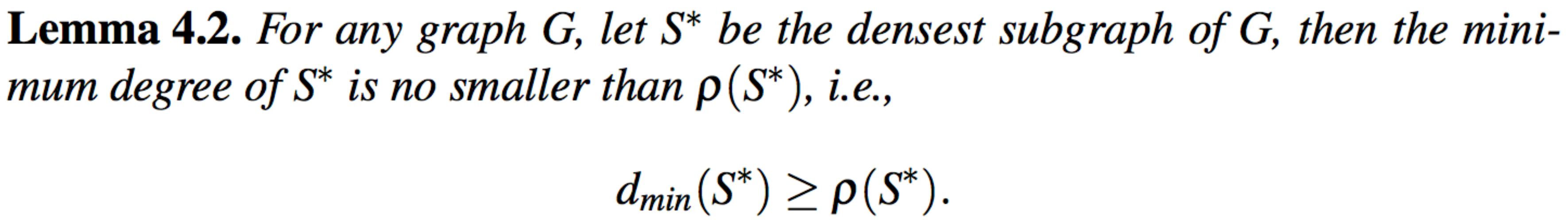
Greedy-based Approximation Algorithm

- Iteratively remove the vertex with the minimum degree from the graph (This is the same as the previous peeling algorithm)
  - We obtain n subgraphs during the process
  - Return the one with the maximum average degree as the result

- Time complexity: O(m)

Greedy-based Approximation Algorithm

- Approximation ratio
  - Density $\rho(S)$ of S: total number of edges divided by total number of vertices (half of the average degree)
  - Upper bound of the maximum density $\rho(S^*)$



Lemma 4.2. *For any graph G, let S^* be the densest subgraph of G, then the minimum degree of S^* is no smaller than $\rho(S^*)$, i.e.,*

$$d_{min}(S^*) \geq \rho(S^*).$$

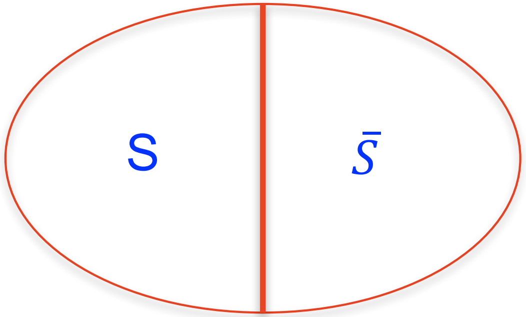
- $\rho(S^*) \leq d_{min}(S^*) \leq \delta(G)$: recall $\delta(G)$ is the maximum value among the minimum vertex degrees of all subgraphs of G

  - Let's look at the $\delta(G)$-core: recall there exists a subgraph with minimum degree $\delta(G)$
    - Its density is at least $\delta(G)/2 \geq \rho(S^*)/2$
  - The approximation ratio of the greedy algorithm is $1/2$

Goldberg's Algorithm for Densest Subgraph

[Goldberg 1984]

- Decision version of the densest subgraph problem: Is there a subgraph S with density larger than λ ?
  - $\rho(S)$: density of S (half of the average degree)
  - $\bar{S}$: vertices of G not in S
  - $E(S,\bar{S})$: edges between S and $\bar{S}$

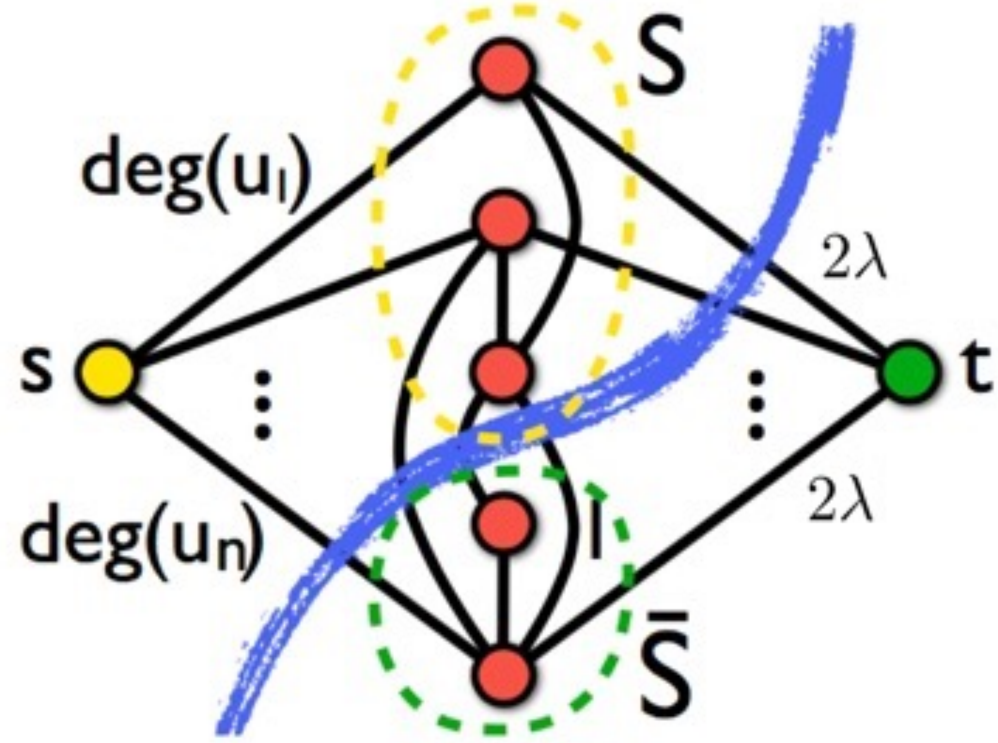


$$
\begin{aligned}
& \rho(S) > \lambda \\
\Longleftrightarrow & 2|E(S)| > 2\lambda|S| \\
\Longleftrightarrow & \sum_{u\in S} d(u) - |E(S,\bar{S})| > 2\lambda|S| \\
\Longleftrightarrow & \sum_{u\in S} d(u) + \sum_{u\in \bar{S}} d(u) - \sum_{u\in \bar{S}} d(u) - |E(S,\bar{S})| > 2\lambda|S| \\
\Longleftrightarrow & \sum_{u\in \bar{S}} d(u) + |E(S,\bar{S})| + 2\lambda|S| < 2|E|
\end{aligned}
$$

Goldberg’s Algorithm for Densest Subgraph

[Goldberg 1984]

There is S s.t. $\rho(S) > \lambda \iff \sum_{u \in \bar{S}} d(u) + |E(S, \bar{S})| + 2\lambda|S| < 2|E|$



G_λ

[Gionis and Tsourakakis 2015]

- $\lambda < \rho(S^*)$ iff the minimum cut of G_λ is of value smaller than 2m
- $\lambda \geq \rho(S^*)$ iff the minimum cut of G_λ is of value exactly 2m

Goldberg's Algorithm for Densest Subgraph

[Goldberg 1984]

- Thus, we can do binary search on λ.
  - When λ is smaller than but very close to $\rho(S^*)$, then the minimum cut of the graph G_λ corresponds to a densest subgraph of G

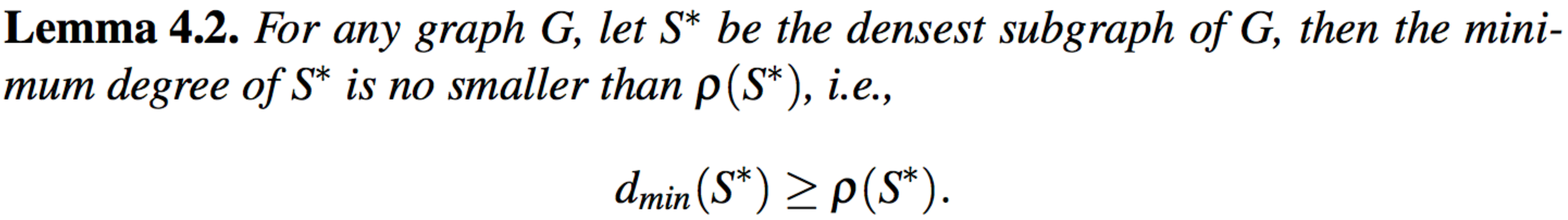
- But λ is a fractional number, when to stop?
  - For any two subgraphs S_1 and S_2 with $\rho(S_1) > \rho(S_2)$, it holds that $\rho(S_1) - \rho(S_2) \geq \frac{1}{n(n-1)}$

- Time complexity of Goldberg's algorithm
  - $O(\log n)$ minimum cut computations, each for a different λ value
  - By using parametric maximum flow techniques, can be implemented to run in $O(\mathrm{n} \cdot \mathrm{m} \cdot \log \frac{n^2}{m})$ time

Data Reduction for Densest Subgraph Computation

- Goldberg's algorithm cannot be directly applied to large graphs, due to the high time complexity

- We can reduce the graph instance for Goldberg's algorithm
  - Real-world graphs are power-law graphs, many vertices are of small degree and thus cannot be in the densest subgraph



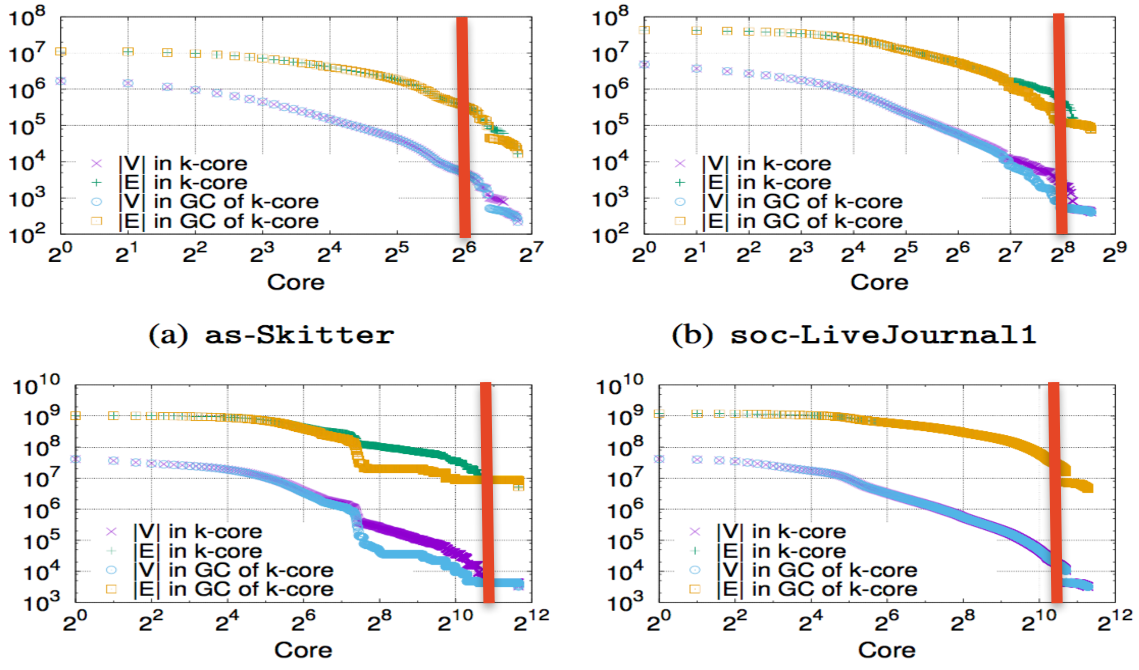
Lemma 4.2. *For any graph G, let S^* be the densest subgraph of G, then the minimum degree of S^* is no smaller than $\rho(S^*)$, i.e.,*

$$d_{min}(S^*) \geq \rho(S^*).$$

  - The density of the $\delta(G)$-core is at least $\delta(G)/2$
  - Thus, we can remove all vertices whose degree are smaller than $\delta(G)/2$

Data Reduction for Densest Subgraph Computation

- Thus, to exactly compute the densest subgraph, we only need to consider the $\delta(G)/2$-core, rather than the entire graph



(a) as-Skitter

(b) soc-LiveJournal1

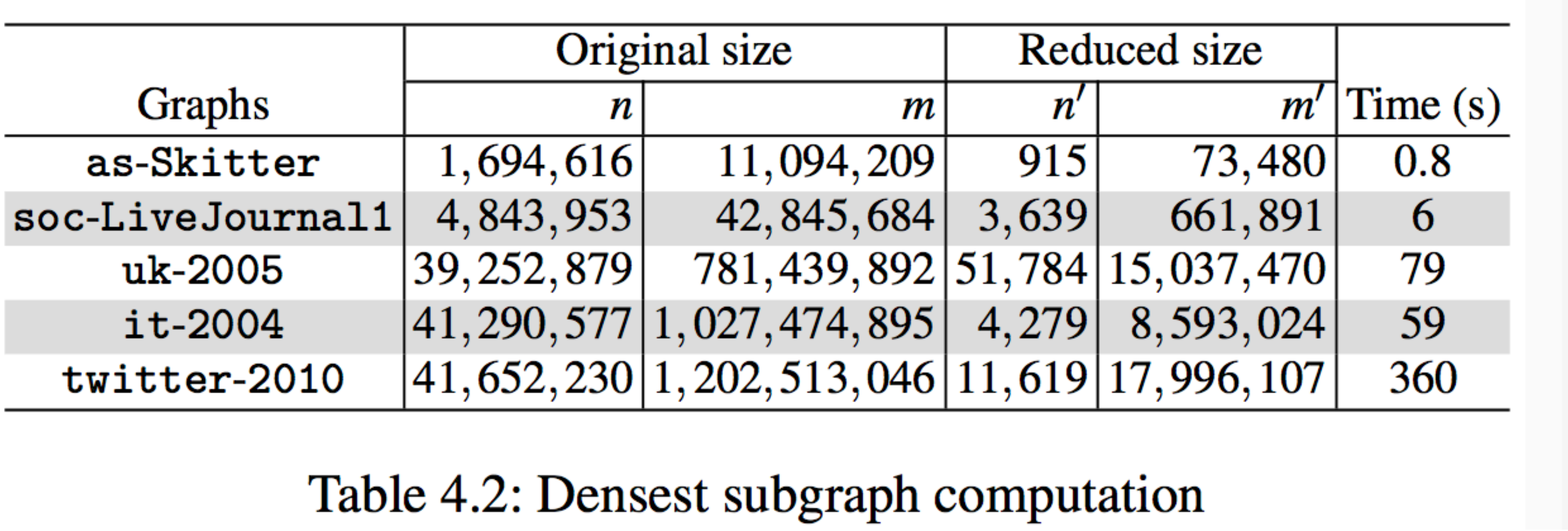
(c) it-2004

(d) twitter-2010

Fig. 3.2: Number of vertices and edges in (Giant Component of) *k*-core (varying *k*)

Data Reduction for Densest Subgraph Computation

- In practice, we can first run the greedy algorithm to get an approximate densest subgraph S, and then run Goldberg's algorithm on the $\lceil \rho(S) \rceil$-core of G



Graphs	Original size		Reduced size		Time (s)
	n	m	n'	m'	
as-Skitter	1,694,616	11,094,209	915	73,480	0.8
soc-LiveJournal1	4,843,953	42,845,684	3,639	661,891	6
uk-2005	39,252,879	781,439,892	51,784	15,037,470	79
it-2004	41,290,577	1,027,474,895	4,279	8,593,024	59
twitter-2010	41,652,230	1,202,513,046	11,619	17,996,107	360

Table 4.2: Densest subgraph computation

Other Works on Densest Subgraph Computation

- The densest subgraph can also be computed by linear programming
  - [Charikar 2000]

- Densest subgraph computation in dynamic graphs
  - [Epasto et al 2015]

- Locally densest subgraph
  - [Qin et al 2015]

- Density-friendly graph decomposition
  - [Danish et al 2017]

Outline

- Background
- Core Decomposition
- Densest Subgraph Computation
- Higher-order Dense Subgraph Computation
- Future Directions

Higher-order Structures

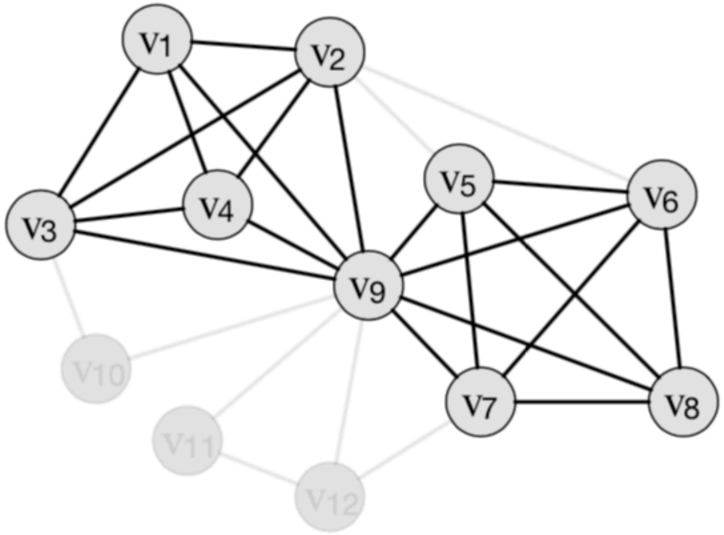
- Previous, we focused on vertices and edges

- Now, let's consider higher-order structures, k-cliques (complete graphs with k vertices), which usually will find denser subgraphs
  - A vertex is a 1-clique
  - An edge is a 2-clique
  - A triangle is a 3-clique

- Higher-order core decomposition
  - Truss decomposition
  - Nucleus decomposition
- Higher-order densest subgraph

Truss Decomposition

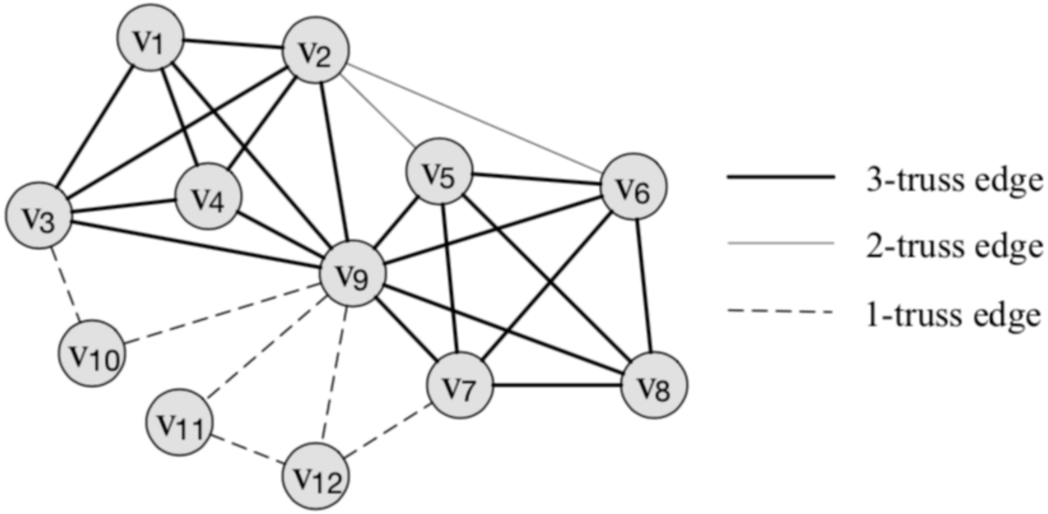
- k-truss: the *maximal* subgraph in which every edge participates in at least k triangles
- k-core: the maximal subgraph in which every vertex participates in at least k edges
- Like k-cores, k-trusses are nested



3-truss

Truss Decomposition

- Truss number *truss(u,v)* of an edge: the largest *k* for which the *k*-truss contains the edge
- Truss decomposition: compute the truss number for each edge
  - *k*-truss is the subgraph induced by all edges with truss numbers at least *k*



Truss decomposition

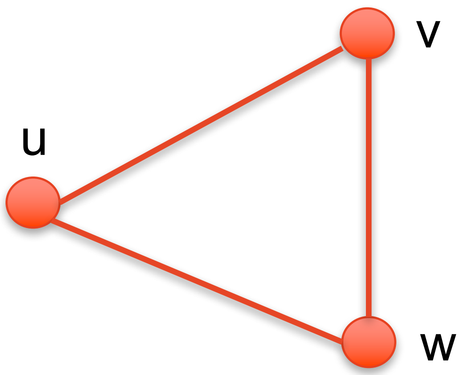
Computing Truss Decomposition

- Extend the peeling algorithm
  - Iteratively remove the edge that participates in the fewest number of triangles
  - How to efficiently compute the number of triangles for each vertex?
  - How to efficiently update the number of triangles after deleting one edge?

  - This needs an efficient triangle enumeration algorithm.

Triangle Enumeration

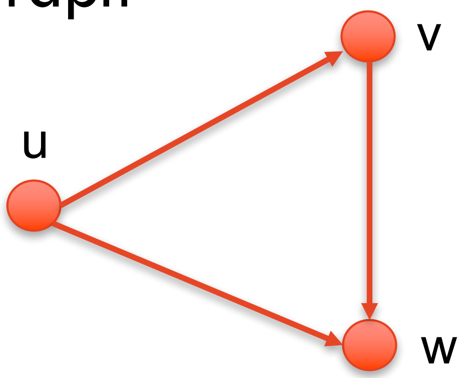
- How to efficiently enumerate all triangles in a graph?

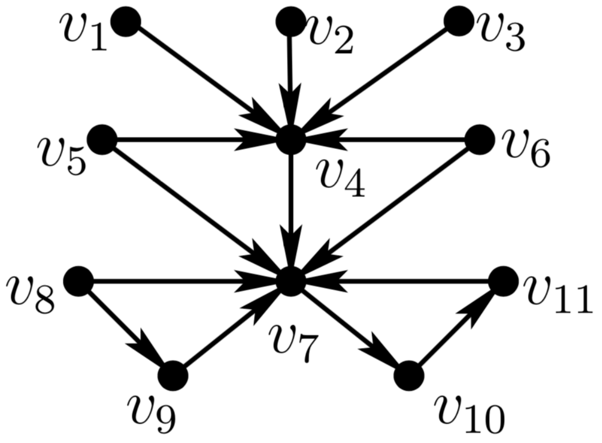


- How about enumerating wedge (v,u,w) and check the existence of edge (v,w)?
  - The time complexity will be $\sum_{u \in V}(d(u))^2$
    - This is higher than $m^{\frac{3}{2}}$, which bounds the maximum number of triangles
    - E.g., consider a star graph

Triangle Enumeration

- We can improve the time complexity by orienting the input graph



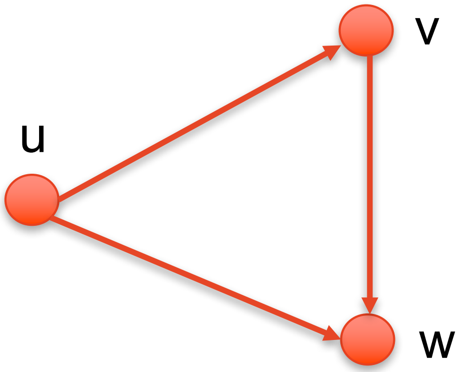


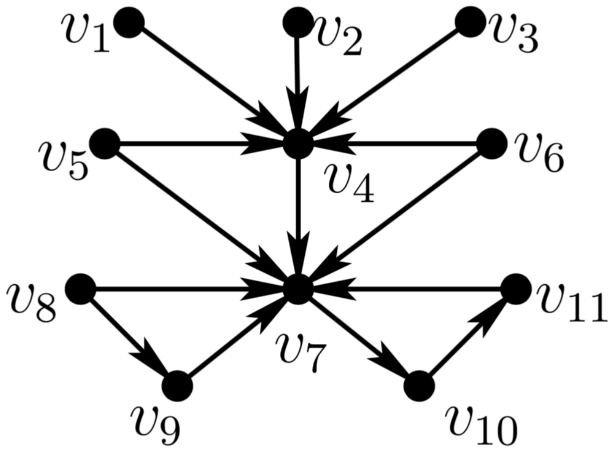
Oriented graph

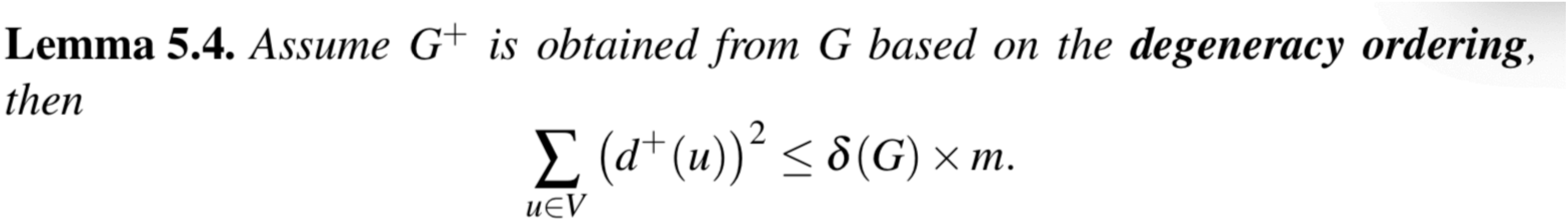
- How about enumerating wedge (v,u,w) and check the existence of edge (v,w)?
  - The time complexity will be $\sum_{u \in V} (d^+(u))^2$
  - This can be small if we orient the graph smartly

Triangle Enumeration

How about enumerating wedge (v,u,w) and check the existence of edge (v,w)?







Lemma 5.4. *Assume G^+ is obtained from G based on the* ***degeneracy ordering****, then*

$$\sum_{u \in V} \left(d^+(u)\right)^2 \leq \delta(G) \times m.$$

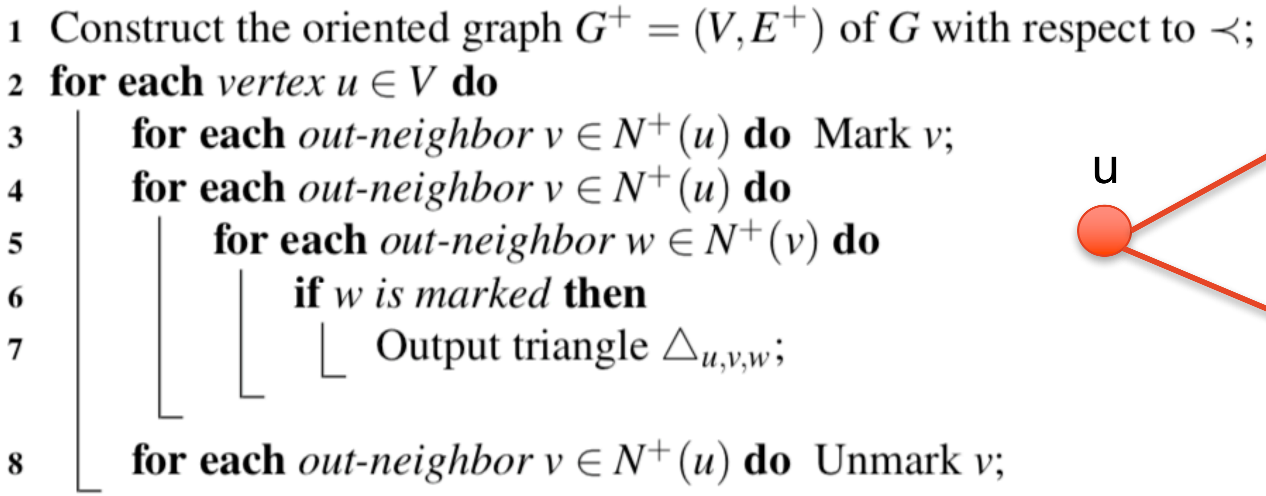
Recall that $\delta(G) \leq \lceil \sqrt{2m+n} \rceil$

The total number of wedges checked is $O(m^{\frac{3}{2}})$

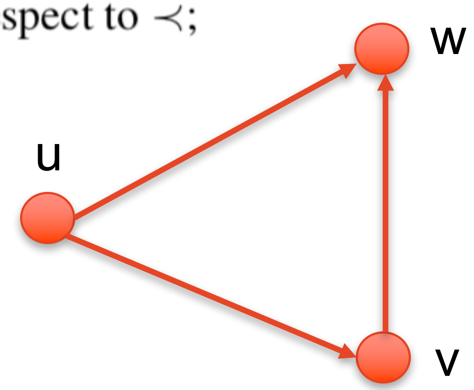
How to check the existence of an edge? Hash table!

Triangle Enumeration

- Hash table has both space and time overhead. Can we avoid the hash-table? Yes!
  - Arrange the edge-existence checking in a smart way.



```
Construct the oriented graph $G^+ = (V, E^+)$ of $G$ with respect to $\prec$;
for each vertex $u \in V$ do
    for each out-neighbor $v \in N^+(u)$ do Mark $v$;
    for each out-neighbor $v \in N^+(u)$ do
        for each out-neighbor $w \in N^+(v)$ do
            if $w$ is marked then
                Output triangle $\triangle_{u,v,w}$;
    for each out-neighbor $v \in N^+(u)$ do Unmark $v$;
```



  - This is to check whether *u* is connected to its 2-hop out-neighbor *w*
  - All such checks for the same *u* are grouped together. Hash table is not needed!
  - The time complexity is no longer $\sum_{u \in V} (d^+(u))^2$
    - but $\sum_{v \in V} (d^+(v) \times d^-(v))$

Triangle Enumeration

```
Construct the oriented graph $G^+ = (V, E^+)$ of $G$ with respect to $\prec$;
for each vertex $u \in V$ do
    for each out-neighbor $v \in N^+(u)$ do Mark $v$;
    for each out-neighbor $v \in N^+(u)$ do
        for each out-neighbor $w \in N^+(v)$ do
            if $w$ is marked then
                Output triangle $\triangle_{u,v,w}$;
    for each out-neighbor $v \in N^+(u)$ do Unmark $v$;
```

Lemma 5.5. *Assume G^+ is obtained from G based on the* ***degeneracy ordering****, then*

$$\sum_{u \in V} d^-(u) \times d^+(u) \leq (2\alpha(G) - 1) \times m.$$

$$\alpha(G) \leq \left\lceil \frac{\sqrt{2m+n}}{2} \right\rceil$$

Lemma 5.3. *Assume G^+ is obtained from G based on the* ***degree*** **decreasing** ***ordering****, then*

$$\sum_{u \in V} d^-(u) \times d^+(u) \leq 2 \times \alpha(G) \times m.$$

As computing degeneracy ordering takes a significant portion of the total time, degree decreasing or increasing ordering is used in practice.

Computing Truss Decomposition

- Extend the peeling algorithm
  - Iteratively remove the edge that participates in the fewest number of triangles
  - How to efficiently compute the number of triangles for each vertex?
    - Enumerate all triangles by the algorithm in previous slide
  - How to efficiently update the number of triangles after deleting one edge (u,v)?
    - Intersect the neighbor-sets of u and v in $\min\{\mathrm{d}(\mathrm{u}), \mathrm{d}(\mathrm{v})\}$ time

Hash table is needed here!

- The algorithm runs in $O(\alpha(G) \times m)$ time
  - $\alpha(G)$ is the arboricity of G, and is small for real graphs

$$\sum_{(u,v)\in E^+} \min\{d(u), d(v)\} \leq 2 \times \alpha(G) \times m$$

Nucleus Decomposition

triangles

- k-(r,s)-nucleus: the maximal union g of s-cliques in G such that for each r-clique C in g, there are at least k s-cliques in g containing C edges
  - k-core is a k-(1,2)-nucleus

    Iteratively remove vertices with fewer than k edges containing the vertex, the remaining edges form the *k*-core
  - k-truss is a k-(2,3)-nucleus

    Iteratively remove edges with fewer than k triangles containing the edge, the remaining triangles form the *k*-truss

Iteratively remove r-cliques with fewer than k s-cliques containing the r-clique, the remaining s-cliques form the *k*-(r,s)-nucleus?

Yes!

Nucleus Decomposition

- Iteratively remove r-cliques with fewer than k s-cliques containing the r-clique, the remaining s-cliques form the *k*-(r,s)-nucleus
- Let's consider the hyper-graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$
  - $\mathbb{V}$ is the set of r-cliques in G
  - $\mathbb{E}$ is the set of s-cliques in G (hyper-edges)
  - Each hyper-edge (s-clique) in $\mathbb{G}$ connects to all r-cliques contained in the s-clique
  - In truss decomposition, r = 2 and s = 3

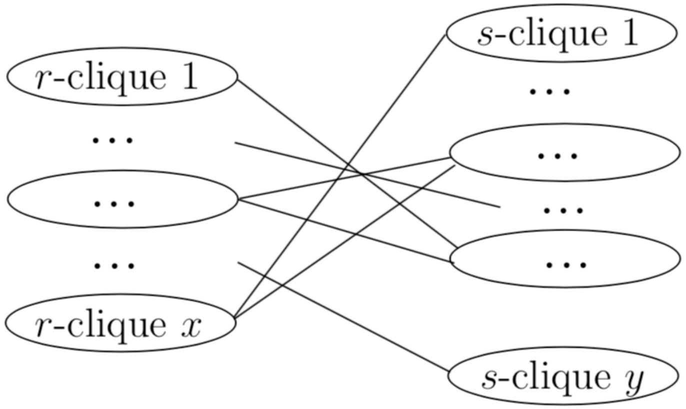


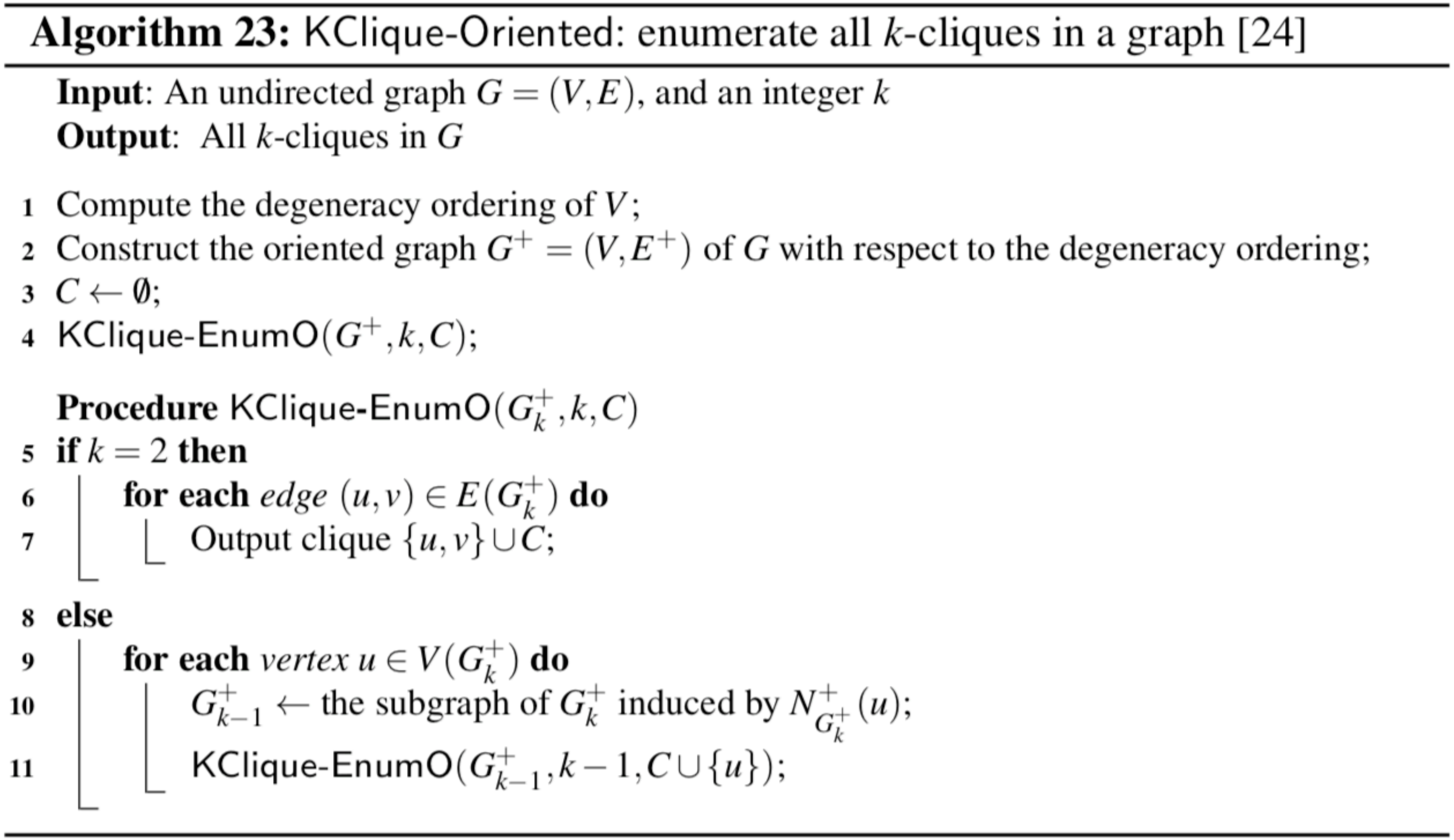
Fig. 5.4: Bipartite graph for nucleus decomposition

k-clique Enumeration

- How to efficiently enumerate k-cliques?
- Extend the graph orientation-based triangle enumeration algorithm
  - Orient the input undirected graph to be a directed graph
  - For each vertex *u* in *G*
    - Enumerate *(k-1)*-cliques in the subgraph of *G* induced by *u*'s out-neighbors

Each *k*-clique of *G* will be enumerated exactly once!

k-clique Enumeration



Algorithm 23: KClique-Oriented: enumerate all k-cliques in a graph [24]

```
Input: An undirected graph $G = (V, E)$, and an integer $k$
Output: All $k$-cliques in $G$

Compute the degeneracy ordering of $V$;
Construct the oriented graph $G^+ = (V, E^+)$ of $G$ with respect to the degeneracy ordering;
$C \leftarrow \emptyset$;
KClique-EnumO$(G^+, k, C)$;

Procedure KClique-EnumO$(G_k^+, k, C)$
if $k = 2$ then
    for each edge $(u, v) \in E(G_k^+)$ do
        Output clique $\{u, v\} \cup C$;
else
    for each vertex $u \in V(G_k^+)$ do
        $G_{k-1}^+ \leftarrow$ the subgraph of $G_k^+$ induced by $N_{G_k^+}^+(u)$;
        KClique-EnumO$(G_{k-1}^+, k-1, C \cup \{u\})$;
```

- All k-cliques can be enumerated in $O(k \times (\alpha(G))^{k-2} \times m)$ total time

Higher-order Densest Subgraph

- *k*-clique densest subgraph: find the subgraph g of G, such that the average number of k-cliques per vertex in g is the largest among all subgraphs of G

- The peeling algorithm can be extended to find a k-approximate k-clique densest subgraph

- The Goldberg's algorithm can be extended to find the k-clique densest subgraph exactly

Outline

- Background
- Core Decomposition
- Densest Subgraph Computation
- Higher-order Dense Subgraph Computation
- Future Directions

Future Directions

- How to do truss decomposition without hash tables?

- What is the relationship between dense subgraphs with different density (average degree) values?

- How to scale up nucleus decomposition and *k*-clique densest subgraphs for large *k* values?

- How to effectively and efficiently incorporate other information (such as attributes, temporal) into dense subgraph computation?

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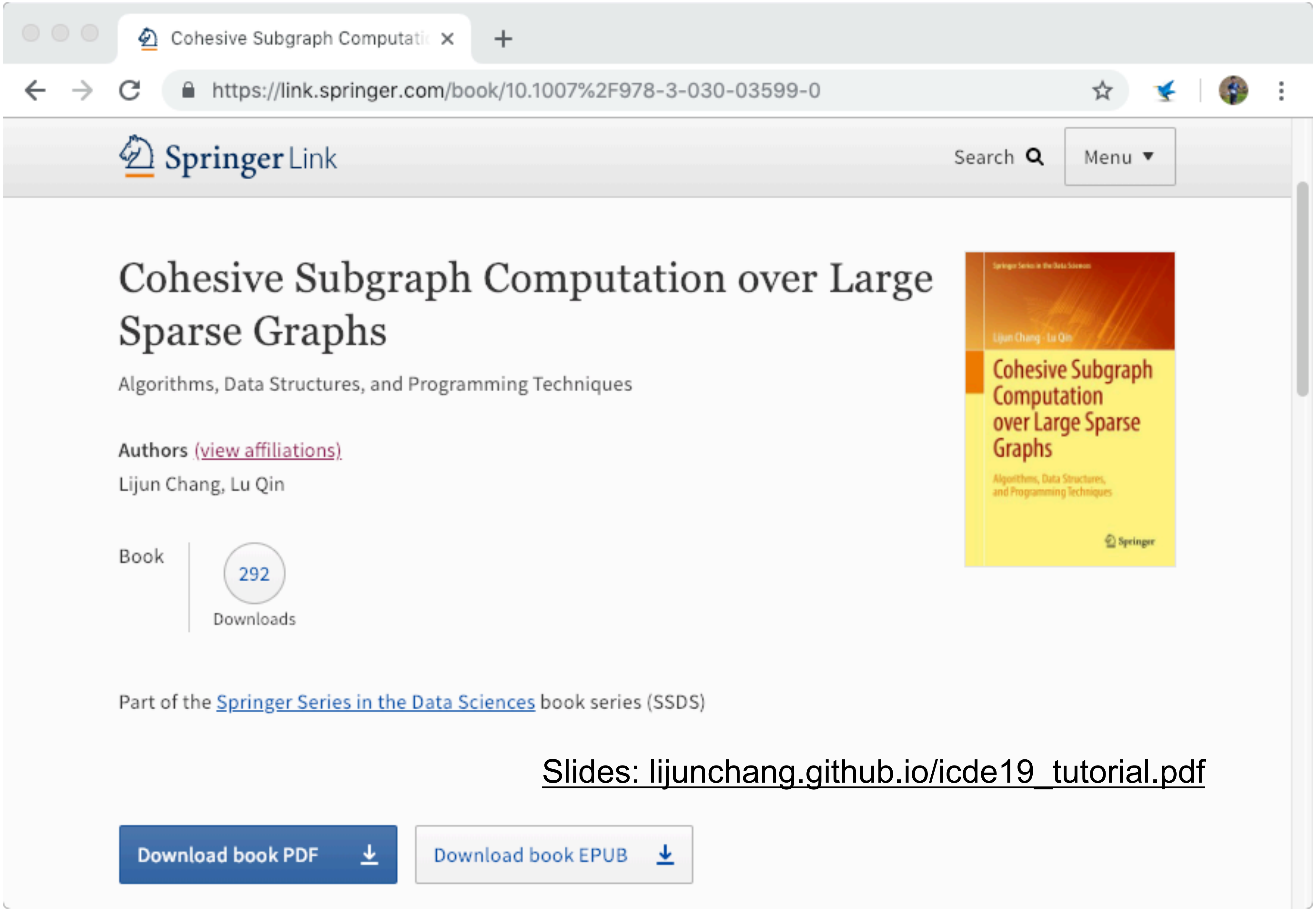

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